TOWARDS MULTI-RIGID BODY LOCALIZATION

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ABSTRACT

In this paper we focus on the relative position and orientation estimation between rigid bodies in an anchorless scenario. Several sensor units are installed on the rigid platforms, and the sensor placement on the rigid bodies is known beforehand (i.e., relative locations of the sensors on the rigid body are known). However, the absolute position of the rigid bodies is not known. We show that the relative localization of rigid bodies amounts to the estimation of a rotation matrix and the relative distance between the centroids of the rigid bodies. We measure all the unknown pairwise distances between the sensors, which we use in a constrained least squares estimator. Furthermore, we also allow missing links between the sensors. The simulations support the developed theory.

***Index Terms*—** Rigid body localization, anchorless network, sensor networks, relative orientation estimation, relative positioning.

1. INTRODUCTION

Automated systems (e.g., underwater vehicles, drones, robots) are designed to closely follow a desired sequence of way points or a trajectory. The knowledge of the current position, that is, localization is crucial for navigating such automated systems. Global positioning system (GPS) is one of the most popular earth-referenced positioning systems used for localization and navigation. However, in many cases of interest, e.g., in underwater applications or indoor environments, the GPS signals are either unavailable or seriously impaired. In such environments, sensor networks provide effective localization solutions.

Localization can be either absolute (e.g., using spatial reference points — anchors) or relative (i.e., without any reference— anchorless). Often, sensors with known absolute positions (i.e., anchors) are deployed in the area nearby the trajectory of the robot for absolute localization. Localization is a well-known problem and has been extensively studied in the past; see [1], [2] for an overview. In emergency situations, an existing infrastructure (e.g., anchors) might not be available or there might not be enough time to set up one. Relative localization in such an anchorless scenario is solved using multi-dimensional scaling [3], [4]. However, relative rigid body localization with prior sensor placement information on the rigid bodies using range measurements has not been studied before.

In this paper, the goal is to localize multiple rigid platforms an anchorless scenario, that is, to localize one robot with respect to the other. In particular, we estimate the relative orientation and translation of one platform with respect to the other. The state-of-the-art orientation estimation relies on an inertial measurement unit (IMU), i.e., a system of sensors comprising of an accelerometer and a gyroscope, which are able to compute the linear and angular motion of the robot. Unfortunately, due to errors in the measurements, an IMU will have drift in velocity and attitude, and thus in the position. Additional expensive sensors can be used to correct these drift errors, but cannot avoid it completely. Thus, whenever GPS signals are not available, or the IMU's drift becomes too large, there is a need for other methods that increase the localization accuracy. Therefore, the proposed framework can be also used as an add-on to correct for the IMU's drift errors.

Recently, it was shown in [5] and [6], that a rigid platform can be localized and tracked using distance measurements. That is, the position and orientation of a rigid body can be estimated using a few sensors installed on the platform and some anchors. Here, we propose an extension of [5] to an anchorless network, where for the sake of exposition we assume only two rigid bodies. More specifically, we estimate the relative angles and translation between them rather than the absolute angles and translation as in [5]. In order to solve the anchorless problem, we adopt a multi-dimensional scaling (MDS)-like approach by making use of only the noisy range measurements among the sensors on the rigid bodies. We propose a constrained least squares estimator, which solves an optimization problem on the Stiefel manifold and an algorithm to compute the relative distance between the centroids of the two rigid bodies. We also account for the missing links (hence, distance measurements) between the sensors. Missing links might occur due to the body geometry and non-availability of a line-of-sight path. Simulations are provided to support the developed theory.

2. PROBLEM FORMULATION

2.1. Modeling

Without loss of generality, consider two rigid bodies, each equipped with N sensors, e.g., installed at the factory. The relative locations of the sensors on the body are assumed to be known. However, the absolute position of the body, or the relative position of one body with respect to the other is not known. The rigid body experiences rotations and translations in each dimension; see the illustration in Fig.1.

Let us introduce the 3-dimensional Stiefel manifold [7], denoted by

$$\mathcal{V}_{3,3} = \{\mathbf{Q} \in \mathbb{R}^{3\times 3} | \mathbf{Q}^\mathsf{T}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^\mathsf{T} = \mathbf{I}_3\}. \tag{1}$$

The absolute initial position of a sensor $n \in \{1, \cdots, N\}$ belonging to the body $i \in \{1, 2\}$ in that reference frame is determined by a 3×1 coordinate vector denoted by $\mathbf{c}_{n,i}$. Thus, the information about the *sensor topology*, namely, how the sensors have been placed on

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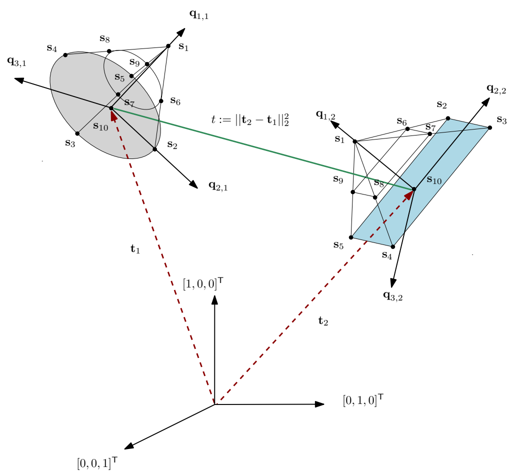


Fig. 1: Illustration of two rigid bodies each equipped with $N = 10$ sensors undergoing a rotation and translation.

each body is determined by $\mathbf{C}_i = [\mathbf{c}_{1,i}, \cdots, \mathbf{c}_{N,i}] \in \mathbb{R}^{3\times N}$ and it is assumed to be perfectly known.

We can express the absolute position of the nth sensor belonging to the body i by $\mathbf{s}_{n,i} \in \mathbb{R}^3$ using the *rigid body transformation* [6]:

$$\mathbf{s}_{n,i} = \mathbf{Q}_i \mathbf{c}_{n,i} + \mathbf{t}_i, \qquad n = 1, 2, \ldots, N, i = 1, 2, \tag{2}$$

where $\mathbf{Q}_i \in \mathcal{V}_{3,3}$ and $\mathbf{t}_i \in \mathbb{R}^3$ are the rotation matrix and translation vector, respectively. The former tells us how the sensor constellation has been rotated in the reference frame while the latter provides the translation of the zero vector in $\mathbb{R}^3$. To consider all the sensors on the body, the N-tuple of vectors $\{\mathbf{s}_{n,i}\}_{n=1}^N$ is stacked into a matrix $\mathbf{S}_i = [\mathbf{s}_{1,i}, \ldots, \mathbf{s}_{N,i}] \in \mathbb{R}^{3\times N}$ which can be expressed as

$$\mathbf{S}_i = \mathbf{Q}_i \mathbf{C}_i + \mathbf{t}_i \mathbf{1}_N^\mathsf{T} = \left[\begin{array}{c|c} \mathbf{Q}_i & \mathbf{t}_i \end{array}\right] \left[\begin{array}{c} \mathbf{C}_i \\ \hline \mathbf{1}_N^\mathsf{T} \end{array}\right], \qquad i = 1, 2. \tag{3}$$

Finally, the absolute position of all the $2N$ sensors can be defined with the $3 \times 2N$ matrix $\mathbf{S} = [\mathbf{S}_1 | \mathbf{S}_2]$ as

$$\mathbf{S} = \underbrace{\left[\begin{array}{c|c} \mathbf{Q}_1 & \mathbf{Q}_2 \end{array}\right]}_{\tilde{\mathbf{Q}} \in \mathbb{R}^{3\times 6}} \underbrace{\left[\begin{array}{c|c} \mathbf{C}_1 & \mathbf{0}_{3\times N} \\ \hline \mathbf{0}_{3\times N} & \mathbf{C}_2 \end{array}\right]}_{\tilde{\mathbf{C}} \in \mathbb{R}^{6\times 2N}} + \underbrace{\left[\begin{array}{c|c} \mathbf{t}_1 & \mathbf{t}_2 \end{array}\right]}_{\tilde{\mathbf{t}} \in \mathbb{R}^{3\times 2}} \underbrace{\left[\begin{array}{c|c} \mathbf{1}_N^\mathsf{T} & \mathbf{0}_N^\mathsf{T} \\ \hline \mathbf{0}_N^\mathsf{T} & \mathbf{1}_N^\mathsf{T} \end{array}\right]}_{\tilde{\mathbf{B}} \in \mathbb{R}^{2\times 2N}}$$

$$= \underbrace{\left[\begin{array}{c|c} \tilde{\mathbf{Q}} & \tilde{\mathbf{t}} \end{array}\right]}_{\mathbf{\Theta}} \underbrace{\left[\begin{array}{c} \tilde{\mathbf{C}} \\ \hline \tilde{\mathbf{B}} \end{array}\right]}_{\mathbf{C}_e}$$

with $\mathbf{\Theta} \in \mathbb{R}^{3\times 8}$ and $\mathbf{C}_e \in \mathbb{R}^{8\times 2N}$ as the unknown transformation matrix and the known augmented topology matrix, respectively. In [6], the aim was to estimate the absolute sensor positions, that is, to estimate the transformation matrix $\mathbf{\Theta}$ using a few reference nodes (i.e., anchors) based on distance measurements between all the sensor-anchor pairs. In contrast, here we focus on determining the relative sensor positions. This is reminiscent of the multidimensional scaling problem, where we estimate only the relative positions of the sensors (i.e., only the constellation of sensors) because of the absence of reference nodes.

The proposed relative rigid body localization problem can be addressed by computing two quantities: the *relative translation* $t \in \mathbb{R}$, which is the Euclidean distance between the two centroids of the rigid bodies and the *relative rotation matrix* $\mathbf{Q} \in \mathcal{V}_{3,3}$. These can respectively be expressed as

$$t = (1/N^2)\|(\mathbf{S}_2 - \mathbf{S}_1)\mathbf{1}_N\|_2^2 = \|\mathbf{t}_2 - \mathbf{t}_1\|_2^2 \in \mathbb{R}, \tag{4a}$$

$$\mathbf{Q} = \mathbf{Q}_1^T \mathbf{Q}_2 \in \mathcal{V}_{3,3}. \tag{4b}$$

An advantage of formulating the problem in this way is that we reduce the number of unknowns from $6N$ corresponding to $\mathbf{S}_1$ and $\mathbf{S}_2$ to 10 unknowns (9 corresponding to $\mathbf{Q}$ and 1 corresponding to t). This gain is because we exploit the rigidity of the body and the prior knowledge of the sensor placements. Note that the absolute sensor positions cannot be recovered, unless a few anchors (at least 3 anchors) are available.

2.2. Measurement model

Assuming that there exists a line-of-sight (LOS) path between all the sensor pairs, each cross-body measurement between the nth and the mth sensor contaminated by additive noise is given as

$$y_{m,n} = \rho_{m,n} + v_{m,n} \qquad n = 1, \cdots, N, \quad m = 1, \cdots, N \tag{5}$$

where $\rho_{m,n} = \|\mathbf{s}_{n,2} - \mathbf{s}_{m,1}\|_2$ is the pairwise distance, and the considered noise process is i.i.d. zero-mean Gaussian with variance σ^2. These observations are considered to be valid as long as the body motion can be neglected during the ranging procedure. The above model is non-linear in $\mathbf{s}_{n,2}, \mathbf{s}_{m,1}$ (thus, in $\mathbf{Q}$ and t). One way to simplify this problem is to *linearize* it, by squaring the measurements, i.e.,

$$y_{m,n}^2 = \rho_{m,n}^2 + n_{m,n} \tag{6}$$

with $\mathbb{E}\{n_{m,n}\} = \sigma^2$ and $\mathbb{E}\{(n_{m,n} - \mathbb{E}\{n_{m,n}\})\} = 4\rho_{m,n}^2\sigma^2 + 2\sigma^4)$ [6]. As can be observed, linearization introduces a bias in the estimates since the noise is not zero-mean any more. If σ^2 is known and high, this bias could be removed. However, for simplicity, we will not consider any noise in the following derivations, unless otherwise stated. To begin with, let us assume that we can measure all the unknown N^2 distances (i.e., there are no missing links). The distances among nodes that belong to the same body are perfectly known, and are not measured. Distances can be collected in a $2N \times 2N$ matrix defined as

$$\mathbf{Y} = \left[\begin{array}{c|c} \mathbf{Y}_1 & \mathbf{Y}_\text{x} \\ \hline \mathbf{Y}_\text{x}^\mathsf{T} & \mathbf{Y}_2 \end{array}\right], \tag{7}$$

where the *known* distance matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are along the block diagonal, and the *noisy* cross-body distance matrix $\mathbf{Y}_\text{x}$ is located in the off-diagonal corners. Due to the knowledge of the sensor topology, the distance matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$ can be computed for $i = 1, 2$ as $[\mathbf{Y}_i]_{m,n} = \|\mathbf{c}_{n,i} - \mathbf{c}_{m,i}\|_2$, $m, n = 1, \cdots, N$. However, the cross-body pairwise distances, that is, the entries of the matrix $\mathbf{Y}_\text{x}$ are measured, hence are noisy. We can collect all the cross-body measurements in (6) in a matrix as

$$\mathbf{Y}_\text{x}^{\odot 2} := \mathbf{Y}_\text{x} \odot \mathbf{Y}_\text{x} = \boldsymbol{\psi}_1 \mathbf{1}_N^\mathsf{T} + \mathbf{1}_N \boldsymbol{\psi}_2^\mathsf{T} - 2\mathbf{S}_1^\mathsf{T}\mathbf{S}_2 \tag{8}$$

with $\boldsymbol{\psi}_i = \left[\|\mathbf{s}_{1,i}\|_2^2, \cdots, \|\mathbf{s}_{N,i}\|_2^2\right]^\mathsf{T} \in \mathbb{R}^{N\times 1}$.

In practice, due to the body geometry, there might be some missing links (i.e., some part of one body will not face the other one). These missing measurements are taken into account via a connectivity matrix $\mathbf{W} \in \mathbb{R}^{N\times N}$ extending (8) to

$$\mathbf{Y}_\text{x}^{\odot 2} \odot \mathbf{W} = \mathbf{W}\text{diag}(\boldsymbol{\psi}_1) + \text{diag}(\boldsymbol{\psi}_2)\mathbf{W} - 2(\mathbf{S}_1^\mathsf{T}\mathbf{S}_2) \odot \mathbf{W}, \tag{9}$$

where the entries of $\mathbf{W}$ are zero if the corresponding link is missing and one otherwise. More general structures for $\mathbf{W}$ could be considered, but here we propose a simplified connectivity matrix, namely

$$\mathbf{W} = \begin{bmatrix} \mathbf{1}_M \mathbf{1}_M^\mathsf{T} & \mathbf{1}_M \mathbf{1}_{N-M}^\mathsf{T} \\ \mathbf{1}_{N-M} \mathbf{1}_M^\mathsf{T} & \mathbf{0}_{N-M} \mathbf{0}_{N-M}^\mathsf{T} \end{bmatrix}. \tag{10}$$

3. PROPOSED ESTIMATORS

In what follows, we will develop estimators for $\mathbf{Q}$ and t from the distance measurements $\mathbf{Y}_\mathrm{x}$.

3.1. Relative rotation matrix estimator

To start with, let us decouple the rotation from the translation. We realize this, as proposed in [8], by eliminating the first two terms in (9) through an orthogonal projection $\mathbf{P}_w \in \mathbb{R}^{N\times N}$ onto the orthogonal complement of $\mathbf{W}$, such that $\mathbf{P}_w\mathbf{W} = \mathbf{W}\mathbf{P}_w = \mathbf{0}$. Let $\tilde{\mathbf{W}}$ be an orthonormal basis for the column span of $\mathbf{W}$, i.e.,

$$\tilde{\mathbf{W}} = \begin{bmatrix} \mathbf{1}_M & \mathbf{1}_M \\ \mathbf{1}_{N-M} & \mathbf{0}_{N-M} \end{bmatrix}. \tag{11}$$

Then, the projection matrix $\mathbf{P}_w = \mathbf{I}_N - \tilde{\mathbf{W}}(\tilde{\mathbf{W}}^\mathsf{T}\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{W}}^\mathsf{T}$ simplifies to

$$\mathbf{P}_w = \begin{bmatrix} \mathbf{\Gamma}_M & \mathbf{0}_M \mathbf{0}_{N-M}^\mathsf{T} \\ \mathbf{0}_{N-M} \mathbf{0}_M^\mathsf{T} & \mathbf{\Gamma}_{N-M} \end{bmatrix}, \tag{12}$$

where $\mathbf{\Gamma}_L = \mathbf{I}_L - L^{-1}\mathbf{1}_L\mathbf{1}_L^\mathsf{T} \in \mathbb{R}^{L\times L}$ is the symmetric centering operator.

Pre- and post-multiplying left and right hand sides of (9) by $\mathbf{P}_w$ and scaling with a factor of $-1/2$, we get

$$\tilde{\mathbf{Y}}_\mathrm{x} = -\frac{1}{2}\mathbf{P}_w(\mathbf{Y}_\mathrm{x}^{\odot 2} \odot \mathbf{W})\mathbf{P}_w = (\underbrace{\mathbf{P}_w\mathbf{S}_1^\mathsf{T}}_{\tilde{\mathbf{S}}_1^\mathsf{T}}\underbrace{\mathbf{S}_2\mathbf{P}_w}_{\tilde{\mathbf{S}}_2}) \odot \mathbf{W}. \tag{13}$$

We now highlight the parts of $\mathbf{S}_1$ and $\mathbf{S}_2$ related to the visible (non-visible) measurements and missing links, namely, $\mathbf{S}_i = [\mathbf{S}_{i,\mathrm{v}} \,|\, \mathbf{S}_{i,\mathrm{nv}}]$, $i = 1, 2$ with entries

$$\mathbf{S}_{i,\mathrm{v}} = \mathbf{Q}_i\mathbf{C}_{i,\mathrm{v}} + \mathbf{t}_i\mathbf{1}_M^\mathsf{T} \in \mathbb{R}^{3\times M}, \tag{14a}$$
$$\mathbf{S}_{i,\mathrm{nv}} = \mathbf{Q}_i\mathbf{C}_{i,\mathrm{nv}} + \mathbf{t}_i\mathbf{1}_{N-M}^\mathsf{T} \in \mathbb{R}^{3\times (N-M)}. \tag{14b}$$

In this way, (13) can be split into the product of two matrices:

$$\tilde{\mathbf{S}}_i = \left[\ \mathbf{S}_{i,\mathrm{v}}\mathbf{\Gamma}_M \ \middle|\ \mathbf{S}_{i,\mathrm{nv}}\mathbf{\Gamma}_{N-M}\ \right] = \left[\ \tilde{\mathbf{S}}_{i,\mathrm{v}} \ \middle|\ \tilde{\mathbf{S}}_{i,\mathrm{nv}}\ \right], \tag{15}$$

where $\tilde{\mathbf{S}}_{i,\mathrm{v}}$ and $\tilde{\mathbf{S}}_{i,\mathrm{nv}}$ are, respectively, the centered sensor submatrices

$$\tilde{\mathbf{S}}_{i,\mathrm{v}} = \mathbf{Q}_i\mathbf{C}_{i,\mathrm{v}} + (\mathbf{t}_i - \mathbf{x}_{i,\mathrm{v}})\mathbf{1}_M^\mathsf{T} \in \mathbb{R}^{3\times M}, \tag{16a}$$
$$\tilde{\mathbf{S}}_{i,\mathrm{nv}} = \mathbf{Q}_i\mathbf{C}_{i,\mathrm{nv}} + (\mathbf{t}_i - \mathbf{x}_{i,\mathrm{nv}})\mathbf{1}_{N-M}^\mathsf{T} \in \mathbb{R}^{3\times (N-M)}. \tag{16b}$$

Here $\mathbf{C}_{i,\mathrm{v}}$ and $\mathbf{C}_{i,\mathrm{nv}}$ contain the first M and the last $N-M$ columns of the sensor topology matrix $\mathbf{C}$, respectively, and $\mathbf{x}_{i,\mathrm{v}}$ and $\mathbf{x}_{i,\mathrm{nv}}$ are the centers of the sensor subconstellations.

By plugging (15) into (13) we get

$$\tilde{\mathbf{Y}}_\mathrm{x} = \left[\begin{array}{c|c} \tilde{\mathbf{S}}_{1,\mathrm{v}}^\mathsf{T}\tilde{\mathbf{S}}_{2,\mathrm{v}} & \tilde{\mathbf{S}}_{1,\mathrm{v}}^\mathsf{T}\tilde{\mathbf{S}}_{2,\mathrm{nv}} \\ \hline \tilde{\mathbf{S}}_{1,\mathrm{nv}}^\mathsf{T}\tilde{\mathbf{S}}_{2,\mathrm{v}} & \mathbf{0}_{N-M}\mathbf{0}_{N-M}^\mathsf{T} \end{array}\right]. \tag{17}$$

We stress the fact that the relative rotation matrix that we are seeking appears in each of the three non-zero entries. Thus, an algorithm that would exploit all the information gathered from the observation matrix $\tilde{\mathbf{Y}}_\mathrm{x}$ would increase the performance in terms of the root mean squared error (RMSE). Alternatively, an easier way to solve it is to select only one of these submatrices, let us say the first one. We can do this by a selection matrix

$$\mathbf{\Phi} = \left[\ \mathbf{I}_M \ \middle|\ \mathbf{0}_{M\times(N-M)}\ \right] \in \mathbb{R}^{M\times N}.$$

Next, we apply an additional projection $\mathbf{\Gamma}_M$ to get rid of all the other terms that do not depend on $\mathbf{Q}$ (i.e., we decouple $\mathbf{Q}_i$ from $\mathbf{t}_i$ for $i = 1, 2$). Pre- and post-multiplying (17), respectively, with $\mathbf{\Gamma}_M\mathbf{\Phi}$ and $\mathbf{\Phi}^\mathsf{T}\mathbf{\Gamma}_M$, we obtain

$$\begin{aligned} \bar{\mathbf{Y}}_\mathrm{x} &= \mathbf{\Gamma}_M\mathbf{\Phi}\tilde{\mathbf{Y}}_\mathrm{x}\mathbf{\Phi}^\mathsf{T}\mathbf{\Gamma}_M \\ &= \left[\ \mathbf{\Gamma}_M\mathbf{C}_{1,\mathrm{v}}^\mathsf{T} \ \middle|\ \mathbf{0}_M\ \right] \left[\begin{array}{c|c} \mathbf{Q}_1^\mathsf{T}\mathbf{Q}_2 & * \\ \hline * & * \end{array}\right] \left[\begin{array}{c} \mathbf{C}_{2,\mathrm{v}}^\mathsf{T}\mathbf{\Gamma}_M \\ \hline \mathbf{0}_M^\mathsf{T} \end{array}\right] \\ &= \mathbf{\Gamma}_M\mathbf{C}_{1,\mathrm{v}}^\mathsf{T}\mathbf{Q}_1^\mathsf{T}\mathbf{Q}_2\mathbf{C}_{2,\mathrm{v}}\mathbf{\Gamma}_M. \end{aligned} \tag{18}$$

So far we have considered a noiseless scenario. If we reintroduce the noise and apply similar operations we obtain an $N\times N$ colored noise matrix $\bar{\mathbf{N}}$. The noisy measurement matrix $\bar{\mathbf{Y}}_\mathrm{x}$ can then be written as

$$\bar{\mathbf{Y}}_\mathrm{x} = \bar{\mathbf{C}}_{1,\mathrm{v}}^\mathsf{T}\mathbf{Q}\bar{\mathbf{C}}_{2,\mathrm{v}} + \bar{\mathbf{N}}, \tag{19}$$

where we have introduced the auxiliary matrices

$$\bar{\mathbf{C}}_{1,\mathrm{v}} = \mathbf{C}_{1,\mathrm{v}}\mathbf{\Gamma}_M \in \mathbb{R}^{3\times M}, \tag{20a}$$
$$\bar{\mathbf{C}}_{2,\mathrm{v}} = \mathbf{C}_{2,\mathrm{v}}\mathbf{\Gamma}_M \in \mathbb{R}^{3\times M}. \tag{20b}$$

The linear model (19) can be simplified to an orthogonal Procrustes problem (OPP) as detailed next. In order to use the OPP theory, we need to get rid of either $\bar{\mathbf{C}}_{1,\mathrm{v}}$ or $\bar{\mathbf{C}}_{2,\mathrm{v}}$, e.g., by multiplying the right-hand side of (19) by $\bar{\mathbf{C}}_{2,\mathrm{v}}^\dagger = \bar{\mathbf{C}}_{2,\mathrm{v}}^\mathsf{T}(\bar{\mathbf{C}}_{2,\mathrm{v}}\bar{\mathbf{C}}_{2,\mathrm{v}}^\mathsf{T})^{-1}$. By doing so, we obtain

$$\check{\mathbf{Y}}_\mathrm{x} = \bar{\mathbf{C}}_{1,\mathrm{v}}^\mathsf{T}\mathbf{Q} + \check{\mathbf{N}}, \tag{21}$$

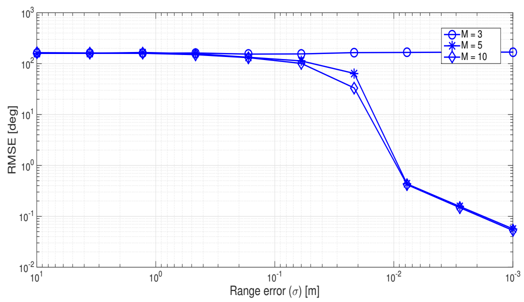
where we defined the following matrices

$$\check{\mathbf{Y}}_\mathrm{x} = \bar{\mathbf{Y}}_\mathrm{x}\bar{\mathbf{C}}_{2,\mathrm{v}}^\dagger \in \mathbb{R}^{M\times 3},$$
$$\check{\mathbf{N}} = \bar{\mathbf{N}}\bar{\mathbf{C}}_{2,\mathrm{v}}^\dagger \in \mathbb{R}^{M\times 3}.$$

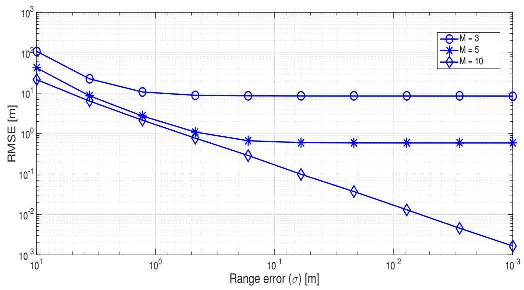
To arrive at (21), we implicitly assume that the wide matrix $\bar{\mathbf{C}}_{2,\mathrm{v}}$ is of full row-rank, i.e., $\mathrm{rank}(\bar{\mathbf{C}}_{2,\mathrm{v}}) = 3$. This, in turn, implies that the sensors span the whole $\mathbb{R}^3$ vector space and this can be guaranteed by avoiding sensor placements that would lead to singularities. However, due to the projection operation (20) $\mathrm{rank}(\bar{\mathbf{C}}_{2,\mathrm{v}}) = M-1$, which means we need at least $M = 4$ sensors on the body. The $\mathbf{Q}$ can be estimated using OPP as

$$\begin{aligned} \hat{\mathbf{Q}}_\mathrm{OPP} = \underset{\mathbf{Q}}{\mathrm{argmin}} \quad & \|\check{\mathbf{Y}}_\mathrm{x} - \bar{\mathbf{C}}_{1,\mathrm{v}}^\mathsf{T}\mathbf{Q}\|_\mathrm{F}^2 \\ \text{subject to} \quad & \mathbf{Q} \in \mathcal{V}_{3\times 3}. \end{aligned} \tag{22}$$

The solution to the above problem can be found by simply performing a singular value decomposition (SVD) [9], i.e., $\hat{\mathbf{Q}}_\mathrm{OPP} = \mathbf{U}\mathbf{V}^\mathsf{T}$, where $\bar{\mathbf{C}}_{1,\mathrm{v}}\check{\mathbf{Y}}_\mathrm{x} =: \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\mathsf{T}$.



(a)



(b)

Fig. 2: RMSE of (2a) relative angles and (2b) relative distance.

3.2. Relative translation estimator

We will now determine the relative displacement between the two rigid bodies, i.e., t. One way to find t, although, not very efficient, is to apply the classical MDS algorithm. In that case we would resolve all the sensor positions (we solve for $6N$ unknowns) to then compute the centroids of the two bodies. This can then be used to obtain t.

To derive a simpler estimator for $t = \|\mathbf{t}_1 - \mathbf{t}_2\|_2^2$, let us consider $\|\mathbf{Y}_{\mathrm{x}}\|_{\mathrm{F}}^2$ (without $\mathbf{W}$ to begin with), which can be expressed as:

$$\|\mathbf{Y}_{\mathrm{x}}\|_{\mathrm{F}}^2 = \mathbf{1}_N^{\mathsf{T}} \mathbf{Y}_{\mathrm{x}}^{\odot 2} \mathbf{1}_N.$$

Using (8) and after some straightforward algebraic operations we can show that

$$\begin{aligned}\|\mathbf{Y}_{\mathrm{x}}\|_{\mathrm{F}}^2 &= N \sum_{n=1}^{N} (\|\mathbf{s}_{n,1}\|_2^2 + \|\mathbf{s}_{n,2}\|_2^2) - 2N^2 \mathbf{t}_1^{\mathsf{T}} \mathbf{t}_2 \\ &= N \sum_{n=1}^{N} (\|\mathbf{c}_{n,1}\|_2^2 + \|\mathbf{c}_{n,2}\|_2^2) + N^2 \|\mathbf{t}_1 - \mathbf{t}_2\|_2^2,\end{aligned}$$

and the exact expression for t is

$$\hat{t} = \frac{1}{N^2} \|\mathbf{Y}_{\mathrm{x}}\|_{\mathrm{F}}^2 - \frac{1}{N} \left(\sum_{n=1}^{N} (\|\mathbf{c}_{n,1}\|_2^2 + \|\mathbf{c}_{n,2}\|_2^2) \right).$$

With missing links instead, we obtain

$$\hat{t} \simeq \frac{1}{M(2N-M)} \|\mathbf{Y}_{\mathrm{x}} \odot \mathbf{W}\|_{\mathrm{F}}^2 - \frac{1}{N} \left(\sum_{n=1}^{N} (\|\mathbf{c}_{n,1}\|_2^2 + \|\mathbf{c}_{n,2}\|_2^2) \right).$$

Note that the approximation error reduces quadratically as M tends to N (i.e., when there are no missing links).

4. NUMERICAL RESULTS

We consider a rectangular-based pyramid and a cone as two rigid bodies. Each rigid body has $N = 10$ sensors installed on it as shown in Fig. 1. In the reference frame, the known topology matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ are respectively given by

$$\mathbf{C}_1 = \begin{bmatrix} 2 & 0 & 0 & -2 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 4 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{bmatrix} \mathrm{m}$$

and

$$\mathbf{C}_2 = \begin{bmatrix} 2 & 2 & 0 & -2 & -2 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 4 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{bmatrix} \mathrm{m}.$$

Both the centers of gravity are set to the origin, so that the relative displacement between the centroids is equal to the distance between the rigid bodies. The two rigid bodies experience a rotation of $\{\psi_1, \theta_1, \phi_1\} = \{20°, -25°, 30°\}$ and $\{\psi_2, \theta_2, \phi_2\} = \{40°, 135°, -75°\}$, respectively. By applying the angular transformation it is then possible to compute the rotation matrices $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}$, from the knowledge of the Euler angles. We use $\mathbf{t}_1 = [1, -5, 4]^{\mathsf{T}}$ m and $\mathbf{t}_2 = [-3, 1, 7]^{\mathsf{T}}$ m. The simulations are averaged over $N_{\mathrm{mc}} = 10^4$ independent Monte-Carlo experiments.
The performance of the proposed estimators is provided in terms of root mean squared error (RMSE) versus the standard deviation σ, of the ranging noise. The RMSE for the relative translation is defined as

$$\sqrt{\frac{1}{N_{\mathrm{mc}}} \sum_{n=1}^{N_{\mathrm{mc}}} |\hat{t}_n - t|_2^2},$$

where $\hat{t}_n$ is the estimated relative translation during the n-th experiment. This is shown in Fig. 2b for different numbers of available links (i.e., for different values of M).

The second performance metric shown in Fig. 2a is related to the RMSE of the relative rotation estimator, which is defined as

$$\sqrt{\frac{1}{N_{\mathrm{mc}}} \sum_{n=1}^{N_{\mathrm{mc}}} \|\mathfrak{F}(\hat{\mathbf{Q}}_{\mathrm{OPP}_n}) - \mathfrak{F}(\mathbf{Q})\|_2^2},$$

with $\hat{\mathbf{Q}}_{\mathrm{OPP}_n}$ the estimates during the n-th experiment. Here, $\mathfrak{F} : \mathcal{V}_{3,3} \to \mathbb{R}^3$ is a non-linear mapping used to compute Euler angles from a rotation matrix. As before, we compute the RMSE of the relative rotation for different numbers of available links (i.e., for different values of M). As we already stated in the remark before, for $M = 3$, due to rank deficiency, the algorithm breaks down. This can be seen in Fig. 2a.

5. CONCLUSIONS

We have proposed a framework for relative localization of two rigid bodies in an anchorless network in this paper. To realize this, we use sensor units that are installed on the rigid platforms, and exploit the known sensor placement on the rigid bodies. We have shown that the relative localization of rigid bodies amounts to the estimation of a rotation matrix (related to the relative angles between the rigid bodies in each dimension) and the relative distance between the centroids of the rigid bodies. Based on distance measurements, we have proposed a constrained least squares estimator for estimating the relative orientation. Furthermore, we have also modeled the missing distance measurements between the sensors across rigid bodies. The simulations show the performance of the proposed estimators.

6. REFERENCES

[1] N. Patwari, J.N. Ash, S. Kyperountas, A.O. Hero, R.L. Moses, and N.S. Correal, "Locating the nodes: cooperative localization in wireless sensor networks," *IEEE Signal Process. Mag.*, vol. 22, no. 4, pp. 54–69, July 2005.

[2] S. Gezici, G.B. Giannakis, H. Kobayashi, A.F. Molisch, H.V. Poor, and Z. Sahinoglu, "Localization via ultra-wideband radios: a look at positioning aspects for future sensor networks," *IEEE Signal Process. Mag.*, vol. 22, no. 4, pp. 70–84, July 2005.

[3] W.S. Torgerson, "Multidimensional scaling: I. Theory and method," *Psychometrika*, vol. 17, no. 4, pp. 401–419, Dec. 1952.

[4] J.A. Costa, N. Patwari, and A.O. Hero, "Distributed weighted-multidimensional scaling for node localization in sensor networks," *ACM Trans. Sens. Networks*, vol. 2, no. 1, pp. 39–64, Feb. 2006.

[5] S. P. Chepuri, G. Leus, and A.-J. Van der Veen, "Rigid Body Localization Using Sensor Networks," *IEEE Trans. Signal Process.*, vol. 62, no. 18, pp. 4911–4924, Sept. 2014.

[6] S. P. Chepuri, A. Simonetto, G. Leus, and A.-J. Van der Veen, "Tracking position and orientation of a mobile rigid body," in *2013 5th IEEE Int. Work. Comput. Adv. Multi-Sensor Adapt. Process.* Dec. 2013, pp. 37–40, IEEE.

[7] L. Eldén and H. Park, "A Procrustes problem on the Stiefel manifold," *Numer. Math.*, vol. 82, no. 4, pp. 599–619, June 1999.

[8] A. Amar, Y. Wang, and G. Leus, "Extending the Classical Multidimensional Scaling Algorithm Given Partial Pairwise Distance Measurements," *IEEE Signal Process. Lett.*, vol. 17, no. 5, pp. 473–476, May 2010.

[9] G.H. Golub and C.F.V. Loan, *Matrix Computations*, JHU Press, 1996.